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Magnetic properties of polychromatic crystals: some theoretical results

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Abstract. The magnetic properties of crystals possessing the symmetry of the 18 polychromatic (3, 4 and 6 coloured) point groups, whose diagrammatic representation was given by Indenbom *et al*, have been studied and the maximum number of independent constants required to describe each of the chosen magnetic properties are enumerated. The existence of the non-vanishing number of independent constants which emerges as a result of this theoretical study—a physical significance for the number of independent constants required to describe a magnetic or physical property and appearing before an irreducible representation μ of the factor group G/H established in the process—suggests investigation by experimental physicists to identify the crystals possessing the symmetry of these polychromatic groups. The results of this study are briefly discussed and summarised.

1. Introduction

It is well known that, though the application of an ordinary symmetry operation on an arrangement of atoms in a point group brings the geometrical structure into coincidence with itself, it may happen that the orientations of some or all of the atomic magnetic moments (spins) might be reversed. In such a case, a further reversal of the affected spins must follow the usual symmetry operation in order to bring the geometrical structure, together with the spins, into complete coincidence with itself. It is in this context that the time reversal (anti-identity) operation R_2 has been introduced to effect the reversal of spins. The introduction of this new operation R_2 increased the number of point groups from 32 to 122 which were broadly divided into three categories—those groups which do not contain R_2 (the 32 ordinary point groups), those which include R_2 explicitly (the 32 grey groups) and those which do not contain R_2 explicitly but contain at least one complementary symmetry operation (Zheludev 1960) usually referred to as the 58 magnetic point groups.

The interpretation of antisymmetry as two-colour symmetry led to the idea of polychromatic symmetry (Belov and Tarkhova 1956). Consequently Indenbom *et al* (1960) derived the 18 polychromatic point groups by introducing the colour changing operation R_n, $n = 3$, 4 or 6 with $R_n^n = \mathrm{E}$, and associated those obtained with the 18 pairs of one-dimensional (1D) complex irreducible representations (IR) of the crystallographic point groups.

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Zamorzaev's (1967) innovative work on quasisymmetry (P-symmetry) groups embraced all the earlier important phenomena of antisymmetry, colour symmetry and cryptosymmetry (Niggli and Wondratschek 1960). The 58 magnetic and 18 polychromatic point groups were shown, with the help of the fundamental quasisymmetry theorem, to be full P-symmetry minor groups with appropriate crystallographic point groups as generators and suitable permutation (cyclic) groups of order 2, 3, 4 or 6 as the group of indices P. A general method of obtaining these different categories of P-symmetry groups as semidirect products was established by Krishnamurty *et al* (1978) where a novel method of associating all the obtained colour symmetry groups with the IR of the generator groups was also suggested employing the idea of allowable irreducible representations (AIR) of the little groups that induce the respective IR of the generator group. Very recently, the idea of composition series that exist among the 32 crystallographic point groups (Lomont 1959) has been explored by this author (Rama Mohana Rao 1985) to bring a host of the earlier results (in respect of the derivation of all the different colour symmetry point groups) onto a common footing by constructing 37 composition series, needed just to generate all the 106 colour symmetry point groups.

The study of the magnetic and physical properties exhibited by the 90 magnetic classes was profitably undertaken in the crystallographic point group and space group studies by several investigators. The character method developed by Bhagavantam (1942) was successfully applied by Bhagavantam and Pantulu (1964) and Bhagavantam (1966) for enumerating the number of independent constants needed for the description of any magnetic property by the 90 magnetic classes. Jahn (1949), Juretschke (1951), Fumi (1952a, b) and Koptsik (1966) have also made notable contributions to the methods of enumeration of physical constants required to describe the various physical properties. A simple and elegant method, based on the group theoretical concept of the factor groups contained in a composition series, of obtaining simultaneously the number of independent constants required to describe a chosen magnetic property by a crystallographic point group and its magnetic variant (if any) was described by Krishnamurty *et al* (1977) with the help of the IR of the factor groups contained in a composition series and through defining the character of a coset.

The polychromatic structure, and hence the polychromatic point groups and space groups, have a useful role to play in many physical applications such as the derivation and description of similarity symmetry point groups and space groups, in the description of stem and layer symmetry groups in higher-dimensional space and in describing the magnetic symmetry of screw (helicoidal) structures, the periods of which do not coincide with the periods of the atomic structures and where the traditional magnetic groups cannot adequately describe the situation (Naish 1963), etc.

In spite of the usefulness of these polychromatic point groups in various applications, the study of the magnetic properties exhibited by the crystal classes possessing the symmetry of these polychromatic groups has not been carried out as far as the author is aware. The possible stationary magnetic moment configurations in crystals cannot be described by the classical and magnetic point groups (Shubnikov groups). On the other hand, the polychromatic and multicolour group apparatus of the P-symmetry structure alone can adequately cover all the aforesaid phenomena, a concrete example of which was provided by Koptsik and Kuzhukeev (1972) with the help of the antiferromagnetic structure of haematite in the range of $253 < T < 948$ K.

A study of the magnetic properties pertaining to the ten crystallographic point groups that generate the 18 polychromatic point groups has already been covered during the study of the 90 magnetic classes and is available in the literature. So the objective of the present paper is to study these 18 applicationally important polychromatic point groups whose possible geometric configuration was provided by Indenbom *et al* (1960) and Bradley and Cracknell (1972) with a view to enumerating the number of independent constants (if they exist) that are required to describe the three magnetic properties.

In § 2, the three magnetic properties that are usually exhibited by the crystals, together with their computed characters, are briefly presented for the sake of completeness. In § 3, the physical significance of the number of constants required to describe a chosen magnetic property and occurring before an IR of the factor group G/H, and those occurring before the total symmetric IR of the variant induced by G, is established through three results. The procedure for obtaining the number of independent constants required to describe a particular magnetic property by the 18 polychromatic variants is explained in respect of piezomagnetism in § 4 and is illustrated for the point groups 3, 4 and 6. The results obtained for the rest of the classes are tabulated (table 2) for all the properties. The nomenclature adopted for the point groups in this paper is that of the Hermann–Maüguin (International) notation and that of the 18 polychromatic classes is due to Indenbom *et al* (1960).

In § 5, the results obtained are discussed briefly. Some suggestions have been made as to the possible studies in which the polychromatic classes that are taken up for the present study are useful. The necessity to invoke these groups in order to describe certain physical situations and phenomena is exemplified.

The occurrence of the non-vanishing number of constants in respect of the three magnetic properties discussed and enumerated for the 18 polychromatic classes in this paper suggests investigations and verification by experimental physicists to find the class of crystals that bear and describe this envisaged symmetry accurately.

2. The three magnetic properties

It has already been established by Bhagavantam and Venkatarayudu (1951) and Bhagavantam (1966) that the physical properties of substances generally represent the relationship between two quantities, each of which may be a scalar, vector or a symmetric tensor of second rank, etc. A physical property is referred to as a magnetic property if one or both of the intersecting physical quantities involve the magnetic field, magnetic induction or magnetic moment as a part thereof. Before the contemplated study is undertaken, let us recall in brief the essential concepts of the three important known magnetic properties usually exhibited by a certain class of crystals: (i) piezomagnetism, (ii) pyromagnetism and (iii) magnetoelectric polarisability

(i) Piezomagnetism is the appearance of a magnetic moment $\boldsymbol{M}(M_i = 1, 2, 3)$ on the application of stress σ. Whereas Tavger (1958), Dzyaloshinskii (1958) and Landau and Lifshitz (1960) have predicted theoretically its existance in magnetic crystals, its occurrence has been experimentally verified and measured in crystals by Borovik-Romanov (1959) in fluorides of cobalt and manganese in the antiferromagnetic state.

(ii) Pyromagnetism is the appearance of a magnetic moment $\boldsymbol{M}(M_i, i = 1, 2, 3)$ on the application of temperature t. The relation between the axial vector $\boldsymbol{M}$ and t can be represented by

$$M_i = \alpha_i t \qquad i = 1, 2, 3 \tag{2.1}$$

where α_i stands for the pyromagnetic tensor that transforms like M_i. The magnetic classes exhibiting this phenomena were identified in certain ferromagnetic classes by Tavger (1958).

(iii) Magnetoelectric polarisability is the production of a magnetic field $\boldsymbol{B}$ (or $\boldsymbol{E}$) on the application of an electric field $\boldsymbol{E}$ (or $\boldsymbol{B}$) in a direction normal to it. If λ_{ij} stands for a second rank tensor representing the magnetoelectric polarisability, then $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by the relation

$$B_i = \sum_j \lambda_{ij} E_j \qquad i, j = 1, 2, 3 \tag{2.2}$$

with the transformation law of λ being taken the same as the product of the representations of $\boldsymbol{E}$ and $\boldsymbol{B}$. The occurrence of this phenomenon was observed and verified by Astrov (1960) and Al'shin and Astrov (1963) in trioxides of chromium and titanium.

The character $\chi(R_\phi)$ corresponding to a symmetry element R_ϕ in the representation is given in terms of these three properties by

$$\chi_a(R_\phi) = (4\cos^2\phi \pm 2\cos\phi)(1 \pm 2\cos\phi) \tag{2.3}$$

$$\chi_b(R_\phi) = 1 \pm 2\cos\phi \tag{2.4}$$

$$\chi_c(R_\phi) = (1 \pm 2\cos\phi)(\pm 1 + 2\cos\phi) \tag{2.5}$$

where the positive or negative signs are to be taken according to whether R_ϕ is a pure rotation or a rotation-reflection through an angle ϕ.

3. The significance of the constants

In what follows, we establish an important result pertaining to the equality of the number of constants required to describe a magnetic (or physical) property and occurring before an IR μ of a factor group G/H and that number required by the corresponding variant of G induced by the IR λ of G for the same property under consideration, where λ of G is engendered by the IR μ of G/H. To this end we invoke the definition of the character of a coset introduced by Krishnamurty *et al* (1977) and prove the following two theorems.

Definition. For any magnetic or physical property, the character of a coset A_iH, where $A_i \in (\mathrm{G}\backslash\mathrm{H})$, in the factor group G/H, is defined as the algebraic sum of the characters of all those elements of the group G that are contained in the coset A_iH in respect of that property divided by the order of the coset.

Theorem 3.1. Let μ be an IR of G/H that engender the IR λ of G. Then the number of times the IR λ of G is contained in the representation for G provided by a physical property is equal to the number of times the IR μ of G/H is contained in the representation for G/H provided by the same physical property.

Proof. Let $\phi(R)$ denote the character provided by the physical property for an element R of the group G. Let $\chi^{(\lambda)}(R)$ denote the character of the same element R of the group G in the IR λ of G. Then by a known property

$$n_\lambda = (1/g) \sum_{R \in \mathrm{G}} \phi(R) \bar{\chi}^{(\lambda)}(R) \tag{3.1}$$

where g is the order of the group G. If $R \in R_i\text{H}$, say $R = R_i h_j$, $h_j \in \text{H}$, then since μ is the IR of G/H which engenders the IR λ of G, one obtains

$$\chi^{(\lambda)}(R) = \chi^{(\mu)}(R_i\text{H}). \tag{3.2}$$

From the definition of the character of a coset introduced earlier, if $\Phi(R_i\text{H})$ denotes the character of the coset $R_i\text{H}$ provided by the chosen physical property, then

$$\begin{aligned}\Phi(R_i\text{H}) &= (1/h) \sum_{h_j \in \text{H}} \phi(R_i h_j) \\ &= (1/h) \sum_{R \in R_i\text{H}} \phi(R)\end{aligned} \tag{3.3}$$

since $R = R_i h_j$.

Therefore

$$\begin{aligned}n_\lambda &= (1/g) \sum_{R_i} \sum_{h_j \in \text{H}} \phi(R_i h_j) \bar{\chi}^{(\lambda)}(R_i h_j) \\ &= (1/g) \sum_{R_i} \sum_{h_j \in \text{H}} \phi(R_i h_j) \bar{\chi}^{(\mu)}(R_i H) \qquad \text{by (3.2)} \\ &= (1/g) \sum_{R_i} \bar{\chi}^{(\mu)}(R_i\text{H}) \sum_{h_j \in \text{H}} \phi(R_i h_j) \\ &= (h/g) \sum_{R_i} \bar{\chi}^{(\mu)}(R_i\text{H}) \Phi(R_i\text{H}) \\ &= (g/h)^{-1} \sum_{R_i} \Phi(R_i\text{H}) \bar{\chi}^{(\mu)}(R_i\text{H}) \\ &= n_\mu .\end{aligned} \tag{3.4}$$

Hence the theorem.

Theorem 3.2. The number of independent constants required to describe a magnetic property and appearing before a 1D complex IR of a crystallographic point group G is equal to the number of magnetic constants needed by the polychromatic variant of G, which is induced by that 1D complex representation of G.

Proof. It has already been established that the method of finding the number of non-vanishing independent constants necessary to describe a magnetic (physical) property of a crystal amounts to obtaining the number n_i that appears before the total symmetric IR of the point group G of order g of the crystal for the same property, following the formula (Bhagavantam and Venkatarayudu 1951)

$$n_i = (1/g) \sum_p h_p \bar{\chi}_p^{(\Gamma_i)} \chi_p^{(\Gamma)}. \tag{3.5}$$

In equation (3.5) h_p is the number of elements in the pth conjugate class, g is the total number of elements of the group G, $\chi_p^{(\Gamma)}$ is the character of the pth conjugate class in the representation Γ and $\chi_p^{(\Gamma_i)}$ is the character of the pth conjugate class in the ith irreducible representation Γ_i of G.

Theorem 3.2 is now established in the light of formula (3.5) with the help of the point group 4. It has already been shown (Rama Mohana Rao 1985) that any one of the IR ^{1}E or ^{2}E of the point group 4 induce the polychromatic group $4^{(4)}$ and that the polychromatic group induced by ^{1}E and ^{2}E of 4 are physically equivalent. The polychromatic group $4^{(4)}$ consists of elements E, $R_4 C_{4z}^+$, $R_4^2 C_{2z}$, $R_4^3 C_{4z}^-$ and the number of constants needed to describe a magnetic property for this variant can be obtained from (3.5) by taking $\chi_p^{(\Gamma_i)}(R) = 1 \forall R \in \text{G}$ which is the characteristic of the total symmetric IR of the polychromatic variant.

Since the polychromatic point group is isomorphic to the point group under consideration, the order of the various conjugate classes in these two groups are the same. Also it was conceived that a complementary symmetry operation R_ϕ (Zheludev 1960) has the same effect as that of R_ϕ but multiplied by -1 as the former reverses the character of a physical property (Indenbom 1960). In an analogous manner, if we contend here that the character of a symmetry operation $R_n R_\phi (R_n^n = \mathrm{E})$ will be the same as that of the character of R_ϕ but multiplied by the nth root of unity, then the n_i in respect of a magnetic property for the total symmetric IR T of the polychromatic variant $4^{(4)}$, for which $h_\rho = 1 \forall \rho$ and $N = 4$, is given from (3.5) by

$$\begin{aligned} n_{(\mathrm{PCG})\mathrm{T}} &= \tfrac{1}{4}\sum_\rho \chi_\rho^{(\Gamma)} \\ &= \tfrac{1}{4}[\chi^{(\Gamma)}(\mathrm{E}) + \chi^{(\Gamma)}(R_4 \mathrm{C}_{4z}^+) + \chi^{(\Gamma)}(R_4^2 \mathrm{C}_{2z}) + \chi^{(\Gamma)}(R_4^3 \mathrm{C}_{4z}^-)] \\ &= \tfrac{1}{4}[\chi^{(\Gamma)}(\mathrm{E}) + \mathrm{i}\chi^{(\Gamma)}(\mathrm{C}_{4z}^+) - 1\chi^{(\Gamma)}(\mathrm{C}_{2z}) - \mathrm{i}\chi^{(\Gamma)}(\mathrm{C}_{4z}^-)]. \end{aligned} \tag{3.6}$$

But $1, \mathrm{i}, -1, -\mathrm{i}$ are respectively the characters of the symmetry operations E, C_{4z}^+, C_{2z}, C_{4z}^- in the IR $^2\mathrm{E}$ of the point group 4 which induces the polychromatic variant $4^{(4)}$.

Hence the value of $n_{(\mathrm{PCG})\mathrm{T}}$ given by equation (3.6) can be expressed as

$$\begin{aligned} n_{(\mathrm{PCG})\mathrm{T}} &= \tfrac{1}{4}[\chi^{(\Gamma)}(E)1 + \chi^{(\Gamma)}(\mathrm{C}_{4z}^+)\mathrm{i} + \chi^{(\Gamma)}(\mathrm{C}_{2z})(-1) + \chi^{(\Gamma)}(\mathrm{C}_{4z}^-)(-\mathrm{i})] \\ &= \tfrac{1}{4}\sum_\rho \chi_\rho^{(\Gamma)} \chi_\rho^{(^2\mathrm{E})} \\ &= \tfrac{1}{4}\sum_\rho h_\rho \chi_\rho^{(\Gamma)} \chi_\rho^{(^2\mathrm{E})}. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7)

$$n_{(\mathrm{PCG})\mathrm{T}} = n_{(^2\mathrm{E})}. \tag{3.8}$$

A similar result can be established in terms of any other polychromatic group generated by the 10 point groups containing complex IR.

The converse of this above result can also be inferred from the fact that there exists a one-to-one correspondence between the polychromatic group induced by a point group G and the pair of 1D complex IR of G that induce it. Thus it may be concluded that the number of independent constants required for the description of a magnetic property in terms of a polychromatic class generated by G can be obtained directly from one of the 1D complex IR of G that induce the polychromatic group. Combining this with the result (3.1) established above, we obtain the following theorem.

Theorem 3.3. The number of independent constants required to describe a magnetic property and appearing before an IR μ of G/H is equal to the number of independent constants required by the corresponding colour group of G induced by the IR λ of G, where the IR λ of G is engendered by the IR μ of G/H.

4. Magnetic constants of the polychromatic classes

Indenbom *et al* (1960) derived the 3-coloured, 4-coloured and 6-coloured point groups (with R_n, $n = 3$, 4 or $6 \ni R_n^n = \mathrm{E}$) using the 10 point groups containing the 1D complex IR as generators. By associating the colour changing operations R_3, R_4 and R_6 with the complex numbers $\omega = \exp(2\pi i/3)$, $i = \exp(2\pi i/4)$and $-\omega^* = \exp(2\pi i/6)$, these authors have shown that the two polychromatic groups associated with the pair of 1D complex representations are mutually isomorphic and thus there are in all 18 polychromatic classes generated by the point groups. These authors also suggested possible diagramatic (geometric) representations for each of the 18 polychromatic point groups, the respective symmetry elements of each of which are provided in table 1 for the sake of reference.

In this section, the magnetic constants of these 18 polychromatic classes, together with the constants required by their generating point groups, are obtained by considering the IR of the appropriate factor groups G/H with the 10 point groups containing the 1D complex IR. The desired constants are obtained by invoking the definition of the character of the coset and utilising the formula (3.5) and also the results established in § 3 (3.1)–(3.3).

To cover the different cases we consider the groups $3^{(3)}$, $4^{(4)}$, and $6^{(3)}/2$ for the magnetic property say, piezomagnetism, the character for which is given by equation (2.1):

$$\chi_{\rho}^{(\Gamma)}(R) = (4\cos^2\phi \pm 2\cos\phi)(1 \pm 2\cos\phi)$$

with the usual notation as explained earlier. The method of enumerating the piezomagnetic constants is explained below.

Table 1. The 18 polychromatic point groups. In column 2, the 18 polychromatic crystal classes are given in the standard notation as given by Indenbom *et al* (1960) and Rama Mohana Rao (1985). In column 3, the 10 crystallographic point groups that generate the 18 polychromatic classes are given in international notation and in column 4 the symmetry elements of the respective polychromatic class are provided. For numbers 16, 17 and 18 the subscript *m* takes the values, *x*, *y* and *z* and the subscript *j* takes the values 1, 2, 3 and 4.

Serial number	C	G	Elements
1	$6^{(6)}$	6	E, $R_6C_6^+$, $R_6^2C_3^+$, $R_6^3C_2$, $R_6^4C_3^-$, $R_6^5C_6^-$
2	$\bar{3}^{(6)}$	$\bar{3}$	E, $R_6S_6^-$, $R_6^2C_3^-$, R_6^3I, $R_6^4C_3^+$, $R_6^5S_6^+$
3	$3^{(3)}/m'$	$3/m$	E, $R_6S_3^-$, $R_6^2C_3^+$, $R_6^3\sigma_h$, $R_6^4C_3^-$, $R_6^5S_3^+$
4	$6^{(3)}$	6	E, $R_3C_6^+$, $R_3^2C_3^+$, C_2, $R_3C_3^-$, $R_3^2C_6^-$
5	$\bar{3}^{(3)}$	$\bar{3}$	E, $R_3S_6^-$, $R_3^2C_3^-$, I, $R_3C_3^+$, $R_3^2S_6^+$
6	$3^{(3)}/m$	$3/m$	E, $R_3S_3^-$, $R_3^2C_3^+$, σ_h, $R_3C_3^-$, $R_3^2S_3^+$
7	$6^{(6)}/m$	$6/m$	E, $R_6C_6^+$, $R_6^2C_3^+$, $R_6^3C_2$, $R_6^4C_3^-$, $R_6^5C_6^-$, R_6^3I, $R_6^4S_3^-$, $R_6^5S_6^-$, σ_h, RS_6^+, $R_6^2S_3^+$
8	$6^{(3)}/m$	$6/m$	E, $R_3C_6^+$, $R_3^2C_3^+$, C_2, $R_3C_3^-$, $R_3^2C_6^-$, I, $R_3S_3^-$, $R_3^2S_6^-$, $R_3S_6^+$, $R_3^2S_3^+$, σ_h
9	$6^{(6)}/m'$	$6/m$	E, $R_6C_6^+$, $R_6^2C_3^+$, $R_6^3C_2$, $R_6^4C_3^-$, $R_6^5C_6^-$, I, $R_6S_3^-$, $R_6^2S_6^-$, $R_6^3\sigma_h$, $R_6^4S_6^+$, $R_6^5S_3^+$
10	$6^{(3)}/m'$	$6/m$	E, $R_6^2C_6^+$, $R_6^4C_3^+$, C_2, $R_6^2C_3^-$, $R_6^4C_6^-$, R_6^3I, $R_6^5S_3^-$, $R_6S_6^-$, $R_6^3\sigma_h$, $R_6^5S_6^+$, $R_6S_3^+$
11	$3^{(3)}$	3	E, $R_3C_3^+$, $R_3^2C_3^-$
12	$4^{(4)}$	4	E, $R_4C_{4z}^+$, $R_4^2C_{2z}$, $R_4^3C_{4z}^-$
13	$\bar{4}^{(4)}$	$\bar{4}$	E, $R_4S_{4z}^-$, $R_4^2C_{2z}$, $R_4^3S_{4z}^+$
14	$4^{(4)}/m$	$4/m$	E, $R_4C_{4z}^+$, $R_4^2C_{2z}$, $R_4^3C_{4z}^-$, R_4^2I, $R_4^3S_{4z}^-$, σ_z, $R_4S_{4z}^+$
15	$4^{(4)}/m'$	$4/m$	E, $R_4C_{4z}^+$, $R_4^2C_{2z}$, $R_4^3C_{4z}^-$, I, $R_4S_{4z}^-$, $R_4^2\sigma_z$, $R_4^3S_{4z}^+$
16	$3^{(3)}/2$	23	E, C_{2m}, $R_3C_{3j}^+$, $R_3^2C_{3j}^-$
17	$\bar{6}^{(3)}/2$	$m3$	E, C_{2m}, $R_3C_{3j}^+$, $R_3^2C_{3j}^-$, I, σ_m, $R_3S_{6j}^-$, $R_3^2S_{6j}^+$
18	$\bar{6}^{(6)}/2$	$m3$	E, C_{2m}, $R_6^2C_{3j}^+$, $R_6^4C_{3j}^-$, R_6^3I, $R_6^3\sigma_m$, $R_6^5S_{6j}^-$, $R_6S_{6j}^+$

It can be seen that the point group 1 requires 18 piezomagnetic constants and the character table of the factor group $3/1 \simeq 3$ is given by

$3/1$	E	C_3^+	C_3^-	n_i
A	1	1	1	6
^{1}E	1	ω	ω^2	6
^{2}E	1	ω^2	ω	6
$\chi_p^{(\Gamma)}$	18	0	0	.

Using formula (3.5), one finds that the point group 3 and its polychromatic variant $3^{(3)}$, which are induced respectively by the IR A and ^{1}E of the factor group $3/1$, require 6 and 6 piezomagnetic constants, respectively.

Similarly for the group $4^{(4)}$ consider the factor group $4/1 \simeq 4$:

$4/1$	E	C_{4z}^+	C_{2z}	C_{4z}^-	n_i
A′	1	1	1	1	4
B	1	−1	1	−1	
^{1}E′	1	−i	−1	i	5
^{2}E′	1	i	−1	−i	5
$\chi_p^{(\Gamma)}$	18	0	−2	0	.

Following the theorem 3.3, it can be inferred that the point group 4 and its polychromatic variant $4^{(4)}$ require 4 and 5 piezomagnetic constants respectively.

To obtain the constants for the polychromatic variant $6^{(3)}/2$ and for the point group 6, consider the normal subgroup 2 of the point group 6 and the factor group $6/2$. Since 2 is a subgroup of index 3-6, the group 6 can be written as the union of the cosets

$$6 = \mathrm{E}\,2 \cup C_3^+\,2 \cup C_3^-\,2.$$

Therefore the character table of $6/2 = 3$ is given by

$6/2$	2	$C_3^+ 2$	$C_3^- 2$	n_i
A″	1	1	1	4
^{1}E″	1	ω	ω^2	2
^{2}E″	1	ω^2	ω	2
$\chi_p^{(\Gamma)}$	8	2	2	.

It has already been observed (Bhagavantam 1966, Krishnamurty *et al* 1977) that the point group 2 requires 8 piezomagnetic constants. So we take 8 as the character of the identity element in $6/2$. The coset $C_3^+ 2$ contains the elements C_3^+ and C_6^- and the character of each of these elements in respect of piezomagnetism is zero and four, respectively. Hence from the definition of the character of the coset introduced earlier, the character of $C_3^+ 2$ is two. Following a similar argument, it can be shown that the character of the coset $C_3^- 2$ is also two. Hence substitution in equation (3.5) shows that the point group 6 and its polychromatic variant $6^{(3)}$ requires 4 and 2 piezomagnetic constants, respectively. Similarly, considering the factor group $6/1$, one can find that 4 piezomagnetic constants are required for the group $6^{(6)}$.

This method can be extended in fact to the rest of the seven point groups that induce the remaining 14 polychromatic variants by suitably choosing the normal subgroup and forming the appropriate factor groups. The number of piezomagnetic

Table 2. Number of constants required to describe the three magnetic properties by the 18 polychromatic classes.

Serial number	Polychromatic class	Number of magnetic constants needed to describe: Piezomagnetism	Pyromagnetism	Magneto electric polarisability
1	$6^{(6)}$	4	1	2
2	$\bar{3}^{(6)}$	0	0	3
3	$3^{(3)}/m'$	4	1	1
4	$6^{(3)}$	2	0	1
5	$\bar{3}^{(3)}$	6	1	0
6	$3^{(3)}/m$	2	0	2
7	$6^{(6)}/m$	0	0	2
8	$6^{(3)}/m$	2	0	0
9	$6^{(6)}/m'$	4	1	0
10	$6^{(3)}/m'$	0	0	1
11	$3^{(3)}$	6	1	3
12	$4^{(4)}$	5	1	2
13	$\bar{4}^{(4)}$	5	1	2
14	$4^{(4)}/m$	0	0	2
15	$4^{(4)}/m'$	5	1	0
16	$3^{(3)}/2$	1	0	1
17	$\bar{6}^{(3)}/2$	1	0	0
18	$\bar{6}^{(6)}/2$	0	0	1

constants required for each of the polychromatic classes (that may exist) are enumerated and tabulated in table 2. A similar procedure can be adopted to find the constants needed to describe the other two magnetic properties in respect of these 10 point groups and their 18 polychromatic classes. As the results pertaining to the 10 generating point groups obtained here are available in the literature, the corresponding results for the 18 polychromatic classes are enumerated and tabulated in table 2 as well.

5. Discussion

In this paper the magnetic properties of the different crystal classes that can adequately describe the symmetry of the 18 polychromatic point groups have been investigated and the number of independent constants required are enumerated in terms of the three chosen magnetic properties.

The 18 polychromatic point groups associated with the 18 pairs of 1D complex representations are nothing but the colour symmetry point groups (with colour value 3, 4 or 6) in which each colour may represent a transformable physical property. These groups can also be viewed as the minor quasisymmetry (P-symmetry) point groups, by finding subgroups of index 3, 4 or 6, to the generating crystallographic point groups containing 1D complex IR, and by restricting $P=\{p, p^2, \ldots, p^m=1\}$ with $m=3$, 4 or 6 where

$$p=\begin{pmatrix} 1 & 2 & 3 & \ldots & m \\ 2 & 3 & 4 & \ldots & 1 \end{pmatrix}.$$

The utility of the colour symmetry groups in the derivation and description of similarity symmetry groups has already been explored and the usefulness of the idea of P symmetry in the description of stem and layer symmetry groups in higher-dimensional space is already appreciated (Zamorzaev 1963, Roman 1959).

By the method outlined here, the constants needed to describe a physical property for all the 18 point groups containing the ID complex IR and the polychromatic variants generated by them can be obtained simultaneously. They need not be calculated separately.

The idea of factor groups occurring for different normal subgroups H for a generator group G can be utilised for the enumeration of magnetic constants in respect of the 10 point groups containing ID complex IR and the 18 polychromatic classes generated by them for all the three magnetic properties.

It can be noted that, in general, the character of the coset A_iH, as defined earlier by Krishnamurty *et al* (1977) and utilised here for a chosen magnetic property in the appropriate factor group G/H, may not be equal to that of the element A_i and its value should not numerically exceed that of H.

An example of physical applications of the colour symmetry groups is that of the magnetic symmetry proposed by Naish (1963). The multiplicative groups constructed by Naish are nothing but the P-symmetry groups. In describing the magnetic symmetry of screw (helicoidal) structures, the periods of which do not coincide with the periods of the atomic structures, the traditional magnetic groups (Shubnikov groups) are not suitable and these polychromatic and multi-colour groups are found to be very useful.

In the light of the results established in this paper on well founded theoretical grounds, we suggest that experimental physicists working in the area of magnetic structures conduct investigations with a view to identifying the possible crystals that exhibit these magnetic properties and which possess the geometric configuration indicated by Indenbom *et al* (1960). That the n_i are non-zero is a positive result produced by the present study and we believe is enough to stimulate such experimental investigation to identify these crystals.

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